Fourier-Bessel Expansions of Electromagnetic Fields in Chiral Cylindrical Structures

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To handle electromagnetic wave propagation in a semi-infinite, perfectly conducting, chiral cylinder with a circular base, on which an harmonic Bessel beam impinges, we present a theory relying on the Fourier-Bessel expansion of electromagnetic fields. The chiral medium is successively described by the Tellegen and Post constitutive relations. Conditions of wave propagation are discussed.

Key words: Fourier-Bessel; Electromagnetic Field; Chiral Cylinder.

1. Introduction

Nanotubes, especially carbon nanotubes, play an important role [1] in outgrowing nano-technology with different applications, particularly in biology, concerning for instance cancer therapy [2]. These devices are used, for instance, to transmit electromagnetic pulses and also as radiation detectors [3] with a sensitivity depending on their chirality; this suggests to investigate the electromagnetic wave propagation in cylinders made of a chiral material.

Such an analysis was performed, a long time ago, in cylindrical chirowaveguides filled with an isotropic lossless material by Engheta and Pelet [4 – 7] using the Tellegen constitutive relations. We are interested here in a somewhat different situation, because we are concerned not with an infinite cylinder but with a semi-infinite cylindrical structure, on which an electromagnetic beam impinges. More precisely, we consider electromagnetic wave propagation in a semi-infinite, perfectly conducting, chiral cylinder in the half-space $z > 0$ with a harmonic Bessel beam incident on its circular base in the $z = 0$ plane; the cylindrical coordinates r, θ, z are used. Then, we prove that the Fourier-Bessel expansion of electromagnetic fields is a suitable tool to handle this problem.

The Fourier-Bessel expansion of an arbitrary function $f(x)$ of a real variable x in terms of the Bessel functions J_m of the first kind of order m is [8, 9]

$$f(x) = \sum_{\nu=0}^{\infty} a_\nu J_m(j_\nu x), \tag{1}$$

in which j_1, j_2, $j_3 \ldots$ are the positive zeroes of the Bessel function $J_m(x)$ arranged in ascending order of magnitude, while the amplitudes a_ν are given by the relation [8, 9]

$$[J_{m+1}(j_\nu)]^2 a_\nu = 2\int_0^1 x f(x) J_m(j_\nu x)\,\mathrm{d}x. \tag{2}$$

The choice of J_m may be somewhat arbitrary but, for instance, Maxwell's equations imply, as shown in Section 2, to work with only J_0 and J_1 for fields symmmetric around x since they do not depend on θ.

To make a comparison with [6] easier, we start this work (Sections 2 – 4) with the Tellegen constitutive relations but, to describe chiral media, there exists a great diversity [10] of such relations among which the Post ones [11] are the most compelling. They follow from the general covariance of electromagnetism under the Lorentz group without any consideration of the medium structure while, for instance, Tellegen put forward his relations on an assumed gyrotropic material. Then the belief, that only covariant quantities have a physical meaning, justifies the use of the Post constitutive relations in Section 5.

To display the formalism of Fourier-Bessel expansions to get the solutions of Maxwell's equations in a chiral cylindrical structure, we first suppose an infinite medium.

2. Electromagnetic Fields in an Infinite Chiral Cylinder

2.1. Maxwell's Equations

With $\exp(-\mathrm{i}\omega t)$ implicit, $c = 1$, and using the cylindrical cordinates r, θ, z the curl Maxwell equations are

$$\begin{aligned} &1/r\partial_\theta E_z - \partial_z E_\theta - \mathrm{i}\omega B_r = 0,\\ &\partial_z E_r - \partial_r E_z - \mathrm{i}\omega B_\theta = 0,\\ &(\partial_r + 1/r)E_\theta - 1/r\partial_\theta E_r - \mathrm{i}\omega B_z = 0, \end{aligned} \tag{3a}$$

$$\begin{aligned} &1/r\partial_\theta H_z - \partial_z H_\theta + \mathrm{i}\omega D_r = 0,\\ &\partial_z H_r - \partial_r H_z + \mathrm{i}\omega D_\theta = 0,\\ &(\partial_r + 1/r)H_\theta - 1/r\partial_\theta H_r + \mathrm{i}\omega D_z = 0, \end{aligned} \tag{3b}$$

and the divergence equations are

$$(\partial_r + 1/r)D_r + 1/r\partial_\theta D_\theta + \partial_z D_z = 0, \tag{4a}$$

$$(\partial_r + 1/r)B_r + 1/r\partial_\theta B_\theta + \partial_z B_z = 0. \tag{4b}$$

We now suppose a chiral cylinder endowed with the Tellegen constitutive relations

$$\boldsymbol{D} = \varepsilon \boldsymbol{E} + \mathrm{i}\xi \boldsymbol{H}, \quad \boldsymbol{B} = \mu \mathbf{H} - \mathrm{i}\xi \boldsymbol{E}, \tag{5}$$

ε, μ, ξ being, respectively, the permittivity, permeability and chirality of the medium.

Taking into account (5), the Maxwell equations become

$$\begin{aligned} &1/r\partial_\theta E_z - \partial_z E_\theta - \mathrm{i}\omega\mu H_r = \omega\xi E_r,\\ &\partial_z E_r - \partial_r E_z - \mathrm{i}\omega\mu H_\theta = \omega\xi E_\theta,\\ &(\partial_r + 1/r)E_\theta - 1/r\partial_\theta E_r - \mathrm{i}\omega\mu H_z = \omega\xi E_z, \end{aligned} \tag{6a}$$

$$\begin{aligned} &1/r\partial_\theta H_z - \partial_z H_\theta + \mathrm{i}\omega\varepsilon H_r = \omega\xi H_r,\\ &\partial_z H_r - \partial_r H_z + \mathrm{i}\omega\varepsilon H_\theta = \omega\xi H_\theta,\\ &(\partial_r + 1/r)H_\theta - 1/r\partial_\theta H_r + \mathrm{i}\omega\varepsilon H_{rz} = \omega\xi H_z, \end{aligned} \tag{6b}$$

and the divergence equations

$$\begin{aligned} &\varepsilon\left[(\partial_r + 1/r)E_r + 1/r\partial_\theta E_\theta + \partial_z E_z\right]\\ &= -\mathrm{i}\xi\left[(\partial_r + 1/r)H_r + 1/r\partial_\theta H_\theta + \partial_z H_z\right],\\ &\mu\left[(\partial_r + 1/r)H_r + 1/r\partial_\theta H_\theta + \partial_z H_z\right]\\ &= \mathrm{i}\xi\left[(\partial_r + 1/r)E_r + 1/r\partial_\theta E_\theta + \partial_z E_z\right] \end{aligned}$$

imply

$$(\partial_r + 1/r)E_r + 1/r\partial_\theta E_\theta + \partial_z E_z = 0, \tag{7a}$$

$$(\partial_r + 1/r)H_r + 1/r\partial_\theta H_\theta + \partial_z H_z = 0. \tag{7b}$$

2.2. Wave Equations

Substituting the expressions of H_r, H_θ of (6a) into the third equation of (6b) gives, with $n^2 = \omega^2\varepsilon\mu$,

$$\begin{aligned} &-\left[\partial_r{}^2 + 1/r\partial_r + 1/r^2\partial_\theta{}^2 + n^2\right]E_z\\ &+ \partial_z\left[(\partial_r + 1/r)E_r + 1/r\partial_\theta E_\theta\right]\\ &= \omega\xi\left[(\partial_r + 1/r)E_\theta - 1/r\partial_\theta E_r + \mathrm{i}\omega\mu H_z\right]. \end{aligned} \tag{8}$$

Taking into account (7a) and the third equation of (6a), respectively, in the second and third square bracket, we get

$$\begin{aligned} &-\left[\partial_r{}^2 + 1/r\partial_r + 1/r^2\partial_\theta{}^2 + \partial_z{}^2 + n^2 + \omega^2\xi^2\right]E_z\\ &= -2\mathrm{i}\omega^2\xi\mu H_z. \end{aligned} \tag{8a}$$

Proceeding similarly with the triplet H_r, H_z, E_θ, leads to

$$\begin{aligned} &-\left[\partial_r{}^2 + 1/r\partial_r - 1/r^2 + \partial_z{}^2 + n^2\right]E_z\\ &+ 1/r\partial_\theta\left[\partial_z E_z - 1/rE_r + \partial_r E_r\right]\\ &= \omega\xi\left[\partial_z E_r - \partial_r E_z + \mathrm{i}\omega\mu H_\theta\right], \end{aligned} \tag{9}$$

which becomes, by substituting (7a) and the second equation of (6a) in the second and third bracket,

$$\begin{aligned} &-\left[\partial_r{}^2 + 1/r\partial_r - 1/r^2 + 1/r^2\partial_\theta{}^2 + \partial_z{}^2 + n^2\right.\\ &\left.+ \omega^2\xi^2\right]E_\theta - 2/r^2\partial_\theta E_r = -2\mathrm{i}\omega^2\xi\mu H_\theta. \end{aligned} \tag{9a}$$

Finally with the triplet H_θ, H_z, E_r, we get

$$\begin{aligned} &-\left[1/r^2\partial_\theta{}^2 + \partial_z{}^2 + n^2\right]E_r + \left[\partial_r\partial_z E_z + 1/r\partial_\theta(\partial_r\right.\\ &\left.+ 1/r)E_\theta\right] = \omega\xi\left[1/r\partial_\theta E_z - \partial_z E_\theta + \mathrm{i}\omega\mu H_r\right]. \end{aligned} \tag{10}$$

Equation (10) transforms into

$$\begin{aligned} &-\left[\partial_r{}^2 + 1/r\partial_r - 1/r^2 + 1/r^2\partial_\theta{}^2 + \partial_z{}^2 + n^2\right.\\ &\left.+ \omega^2\xi^2\right]E_r + 2/r^2\partial_\theta E_z = -2\mathrm{i}\omega^2\xi\mu H_r \end{aligned} \tag{10a}$$

with (7a) and the first equation of (6a) substituted into the second and third square bracket of (10).

From now on, we assume that the electromagnetic field is symmetric around the z-axis of the cylinder which implies that $\partial_\theta\{\boldsymbol{E},\boldsymbol{H}\} = 0$. Then, using the notations

$$\begin{aligned} &\Delta_0 = \partial_r{}^2 + 1/r\partial_r + \partial_z{}^2,\\ &\Delta_1 = \partial_r{}^2 + 1/r\partial_r - 1/r^2 + \partial_z{}^2, \end{aligned} \tag{11}$$

the wave equations (8a), (9a), (10a) reduce to

$$\begin{aligned} &\left(\Delta_0 + m^2\right)E_z - 2\mathrm{i}\omega^2\xi\mu H_z = 0,\\ &\left(\Delta_1 + m^2\right)\{E_r, E_\theta\} - 2\mathrm{i}\omega^2\xi\mu\{H_r, H_\theta\} = 0, \end{aligned} \tag{12}$$

and

$$m^2 = n^2 + \omega^2\xi^2 = \omega^2\left(\varepsilon\mu + \xi^2\right). \tag{12a}$$

Now, changing $\{\boldsymbol{E},\boldsymbol{H}\}$ into $\{\boldsymbol{H},\boldsymbol{E}\}$ and μ into $-\varepsilon$ transforms Maxwell's equations (6a), (7a) into (6b), (7b). This transformation applied to (12) gives

$$\begin{aligned}&\left(\Delta_0+m^2\right)H_z+2\mathrm{i}\omega^2\xi\varepsilon E_z=0,\\&\left(\Delta_1+m^2\right)\{H_r,H_\theta\}+2\mathrm{i}\omega^2\xi\varepsilon\{E_r,E_\theta\}=0.\end{aligned}\tag{13}$$

2.3. *Elementary Solutions of Wave and Maxwell's Equations*

We introduce the field $\boldsymbol{Q}=\sqrt{\varepsilon}\boldsymbol{E}+\mathrm{i}\sqrt{\mu}\boldsymbol{H}$. Then, we get from (12) and (13) the wave equations

$$\left(\Delta_0+q^2\right)Q_z=0,\quad\left(\Delta_1+q^2\right)\{Q_r,Q_\theta\}=0,\tag{14}$$

$$q^2=m^2-2\omega^2\xi\sqrt{\varepsilon\mu}=\omega^2\left(\sqrt{\varepsilon\mu}-\xi\right)^2\tag{14a}$$

with the elementary solutions

$$\begin{aligned}&Q_z=A_zJ_0\left(q_rr\right)\exp\left(\mathrm{i}q_zz\right),\\&\{Q_r,Q_\theta\}=\{A_r,A_\theta\}J_1\left(q_rr\right)\exp\left(\mathrm{i}q_zz\right),\end{aligned}\tag{15}$$

in which ${q_r}^2+{q_z}^2=q^2$, J_0, J_1 are the Bessel functions of the first kind of order zero and one, respectively, while A_z, A_θ, A_r are arbitrary amplitudes; q_r, q_z can be complex.

To get from (15) the solutions of Maxwell's equations, we write the amplitudes

$$\boldsymbol{A}=\sqrt{\varepsilon}\boldsymbol{M}+\mathrm{i}\sqrt{\mu}\boldsymbol{N}.\tag{16}$$

Then, according to (15) and to the definition of the $\boldsymbol{Q}$-field,

$$\begin{aligned}&\{E_z,H_z\}=\{M_z,N_z\}J_0\left(q_rr\right)\exp\left(\mathrm{i}q_zz\right),\\&\{E_r,H_r\}=\{M_r,N_r\}J_1\left(q_rr\right)\exp\left(\mathrm{i}q_zz\right),\\&\{E_\theta,H_\theta\}=\{M_\theta,N_\theta\}J_1\left(q_rr\right)\exp\left(\mathrm{i}q_zz\right).\end{aligned}\tag{17}$$

Substituting (17) into Maxwell's equations (6a), (6b), (7a), (7b) and using the properties of the Bessel functions

$$\begin{aligned}&\partial_rJ_0\left(q_rr\right)=-q_rJ_1\left(q_rr\right),\\&\left(\partial_r+1/r\right)J_1\left(q_rr\right)=q_rJ_0\left(q_rr\right),\end{aligned}\tag{18}$$

gives the following set of equations on the amplitudes M, N:

$$\begin{aligned}&-\mathrm{i}q_zM_\theta-\mathrm{i}\omega\mu N_r=\omega\xi M_r,\\&\mathrm{i}q_zM_r+q_rM_z-\mathrm{i}\omega\mu N_\theta=\omega\xi M_\theta,\\&q_rM_\theta-\mathrm{i}\omega\mu N_z=\omega\xi M_z,\end{aligned}\tag{19a}$$

$$\begin{aligned}&-\mathrm{i}q_zN_\theta+\mathrm{i}\omega\varepsilon M_r=\omega\xi N_r,\\&\mathrm{i}q_zN_r+q_rN_z+\mathrm{i}\omega\varepsilon M_\theta=\omega\xi N_\theta,\\&q_rN_\theta+\mathrm{i}\omega\varepsilon M_z=\omega\xi N_z,\end{aligned}\tag{19b}$$

and from the divergence equations:

$$q_rM_r+\mathrm{i}q_zM_z=0,\tag{20a}$$

$$q_rN_r+\mathrm{i}q_zN_z=0.\tag{20b}$$

From the third equations of the sets (19a), (19b) and from (20a), (20b) we get the amplitudes M_r, M_θ, N_r, N_θ on terms of M_z, N_z:

$$\begin{aligned}&q_rM_\theta=\mathrm{i}\omega\mu N_z+\omega\xi M_z,\\&q_rM_r=-\mathrm{i}q_zM_z,\end{aligned}\tag{21a}$$

$$\begin{aligned}&q_rN_\theta=-\mathrm{i}\omega\varepsilon M_z+\omega\xi N_z,\\&q_rN_r=-\mathrm{i}q_zN_z.\end{aligned}\tag{21b}$$

It can be seen that the first two equations in (19a), (19b) are identically satisfied with (21a), (21b). So, the elementary solutions of Maxwell's equations depend on two arbitrary amplitudes M_z, N_z, as expected.

3. Wave Propagation in an Infinite, Perfectly Conducting, Chiral Cylinder

We consider in an infinite, perfectly conducting, chiral cylinder with the radius a along the z-axis, the propagation of an electromagnetic wave $\{\boldsymbol{E},\boldsymbol{H}\}$, symmetric with respect to z and consequently independent on the angle θ.

We first remark that the wave equations (14) have as solutions the Fourier-Bessel series [8, 9]

$$\boldsymbol{Q}(r,z)=\sum_{\nu=1}^{\infty}\boldsymbol{Q}_\nu(r,z)\tag{22}$$

with

$$\begin{aligned}&Q_{z,\nu}=A_{z,\nu}J_0\left(j_\nu r/a\right)\exp\left(\mathrm{i}\alpha_\nu z\right),\\&\{Q_{r,\nu},Q_{\theta,\nu}\}=\{A_{r,\nu},A_{\theta,\nu}\}J_1\left(l_\nu r/a\right)\exp\left(\mathrm{i}\beta_\nu z\right),\end{aligned}\tag{23}$$

in which j_ν, l_ν, are, respectively, the positive zeroes of the Bessel functions J_0, J_1 arranged in ascending order of magnitude, while, with q^2 given by (14a),

$$\begin{aligned}&\alpha_\nu=\left(q^2-{j_\nu}^2/a^2\right)^{1/2},\\&\beta_\nu=\left(q^2-{l_\nu}^2/a^2\right)^{1/2},\end{aligned}\tag{23a}$$

$\boldsymbol{A}_\nu$'s are arbitrary amplitudes.

We now have to look for those Fourier-Bessel series, which are solutions of the Maxwell equations (6a), (6b), (7a), (7b) and satisfying the boundary conditions imposed by the continuity of the electromagnetic field on the surface $r = a$ of the perfectly conducting cylinder:

$$E_z(a,z) = 0, \tag{24a}$$

$$E_\theta(a,z) = 0, \tag{24b}$$

$$B_r(a,z) = 0. \tag{24c}$$

The boundary conditions (24a), (24b) are fulfilled for each mode

$$E_{z,\nu} = A_{z,\nu} J_0\,(j_\nu r/a) \exp\,(\mathrm{i}\alpha_\nu z)\,, \tag{25a}$$

$$E_{\theta,\nu} = A_{\theta,\nu} J_1\,(l_\nu r/a) \exp\,(\mathrm{i}\beta_\nu z)\,, \tag{25b}$$

and we get at once from the first equation of (3a)

$$\mathrm{i}\omega B_{r,\nu} = -\partial_z E_{\theta,\nu} = -i\beta_\nu A_{\theta,\nu} J_1(l_\nu r/a) \exp(\mathrm{i}\beta_\nu z), \tag{26}$$

implying the third boundary condition (24c).

Then, using the relations (18) satisfied by the Bessel functions, the third equation of (6a) gives

$$\mathrm{i}\omega\mu H_{z,\nu} = (\partial_r + 1/r) E_{\theta,\nu} - \omega\xi E_{z,\nu}. \tag{27}$$

Substituting (25) into (27) leads to

$$\begin{aligned} \mathrm{i}\omega\mu\, H_{z,\nu} &= (l_\nu/a)\, A_{\theta,\nu} J_0\,(l_\nu r/a)\, \exp\,(\mathrm{i}\beta_\nu z) \\ &\quad - \omega\xi\, A_{z,\nu} J_0\,(j_\nu r/a)\, \exp\,(\mathrm{i}\alpha_\nu z)\,. \end{aligned} \tag{27a}$$

Now, taking into account (25a), we get, from the divergence equation (7a),

$$\begin{aligned} E_{r,\nu} &= -\mathrm{i}\alpha_\nu A_{z,\nu} \left[(\partial_r + 1/r)^{-1} J_0\,(j_\nu r/a)\right] \exp\,(\mathrm{i}\alpha_\nu z) \\ &= -\mathrm{i}\alpha_\nu\,(a/j_\nu) A_{z,\nu}\,[J_1\,(j_\nu r/a)] \exp\,(\mathrm{i}\alpha_\nu z)\,. \end{aligned} \tag{28}$$

According to the Tellegen relations (5), $\mu H_{r,\nu} = B_{r,\nu} + \mathrm{i}\xi\, E_{r,\nu}$, and substituting (26), (28) into this last relation gives

$$\begin{aligned} &\omega\mu H_{r,\nu} = -\beta_\nu A_{\theta,\nu} J_1\,(l_\nu r/a)\, \exp\,(\mathrm{i}\beta_\nu z) \\ &+ \mathrm{i}\omega\xi\,\alpha_\nu\,(a/j_\nu)\,[A_{z,\nu} J_1\,(j_\nu r/a)] \exp\,(\mathrm{i}\alpha_\nu z)\,. \end{aligned} \tag{29}$$

Finally, for the component $H_{\theta,\nu}$, we get from the third equation of (6b)

$$H_{\theta,\nu} = -(\partial_r + 1/r)^{-1}\,[\mathrm{i}\omega\varepsilon\, E_{z,\nu} - \omega\xi\, H_{z,\nu}]\,, \tag{30}$$

and according to (25a) and (27a)

$$\begin{aligned} \mathrm{i}\omega\varepsilon\, E_{z,\nu} &= \mathrm{i}\omega\varepsilon\, A_{z,\nu} J_0\,(j_\nu r/a) \exp\,(\mathrm{i}\alpha_\nu z)\,, \\ \omega\xi\, H_{z,\nu} &= \mathrm{i}\xi\,(l_\nu/\mu a)\, A_{\theta,\nu} J_0\,(l_\nu r/a) \exp\,(\mathrm{i}\beta_\nu z) \\ &\quad + \mathrm{i}\omega\xi^2/\mu\, A_{z,\nu} J_0\,(j_\nu r/a) \exp\,(\mathrm{i}\alpha_\nu z)\,. \end{aligned} \tag{30a}$$

Substituting (30a) into (30) and using the second relation of (18) gives

$$\begin{aligned} &H_{\theta,\nu} = -\mathrm{i}\omega\,(a/\mu j_\nu)\left(\varepsilon\mu - \xi^2\right) A_{z,\nu} J_1\,(j_\nu r/a) \\ &\cdot \exp\,(\mathrm{i}\alpha_\nu z) + \mathrm{i}\xi/\mu A_{\theta,\nu} J_1\,(l_\nu r/a) \exp\,(\mathrm{i}\beta_\nu z)\,. \end{aligned} \tag{31}$$

By this it is achieved to determine the ν-th Fourier-Bessel mode depending on the two amplitudes $A_{z,\nu}$ and $A_{\theta,\nu}$.

4. Wave Propagation in a Semi-Infinite, Perfectly Conducting, Chiral Cylinder

We now consider a semi-infinite, perfectly conducting, chiral cylinder with a circular base in the $z = 0$ plane, on which, from a medium with permittivity ε_0 and permeability μ_0, an harmonic Bessel wave symmetric with respect to z impinges.

The components of this incident field are obtained from (17) and (21b) with $\xi = 0$, while ε, μ are changed into ε_0, μ_0, and q^2 into $\omega^2\varepsilon_0\mu_0$, so that ${q_r}^2 + {q_z}^2 = \omega^2\varepsilon_0\mu_0$. So

$$\begin{aligned} &q_r {M^i}_\theta = \mathrm{i}\omega\mu_0 {N^i}_z, \quad q_r {N^i}_\theta = -\mathrm{i}\omega\varepsilon_0 M^i_z, \\ &q_r M^i_r = -\mathrm{i}\, q_z M^i_z, \quad q_r N^i_r = -\mathrm{i}\, q_z N^i_z. \end{aligned} \tag{32}$$

Now, as shown in the previous two sections, only the z and θ components of the electric field are necessary to determine the electromagnetic field, and, according to (17), the components E^i_z, E^i_θ, of the incident field at $z = 0$ are, with the known amplitudes M^i_z, M^i_θ,

$$E^i_z(r,0) = M^i_z J_0(q_r r), \;\; E^i_\theta(r,0) = M^i_\theta J_1(q_r r). \tag{33}$$

We also have, according to (22) and (25) on the base of the cylinder,

$$\begin{aligned} E_z(r,0) &= \sum_{\nu=1}^{\infty} A_{z,\nu} J_0\,(j_\nu r/a)\,, \\ E_\theta(r,0) &= \sum_{\nu=1}^{\infty} A_{\theta,\nu} J_1\,(l_\nu r/a)\,. \end{aligned} \tag{34}$$

Then matching (33) and (34) gives the amplitudes $A_{z,\nu}$, $A_{\theta,\nu}$ of the Fourier-Bessel expansions through the relation (2). Making successively $f(r) = M^i_z J_0(q_r r)$ and

$f(r) = M_\theta^i J_1(q_r r)$ we get

$$\begin{aligned}[][J_1(j_\nu)]^2 A_{z,\nu} &= 2M_z^i \int_0^1 x J_0(q_r a x) J_0(j_\nu x)\,\mathrm{d}x,\\ [J_2(l_\nu)]^2 A_{\theta,\nu} &= 2M_\theta^i \int_0^1 x J_1(q_r a x) J_1(l_\nu x)\,\mathrm{d}x.\end{aligned} \tag{35}$$

A simple numerical integration supplies these amplitudes and achieves to determine the electromagnetic field inside the semi-infinite cylinder since it depends only on the amplitudes $A_{z,\nu}, A_{\theta,\nu}$.

From a mathematical point of view, all these calculations are formal since nothing is known on the convergence of the infinite series (22). But this does not generate a real difficulty, as long as one is interested in propagating waves, because there exists a positive integer ν_0 such as for $\nu > \nu_0$; the zeroes j_ν, l_ν, according to (23a) make α_ν, β_ν purely imaginary: the corresponding modes do not propagate becoming evanescent waves.

For instance, with $q = 2\pi/\lambda$, α_ν real implies $\lambda/a < 2\pi/j_\nu$, the first two zeroes of J_0 are $j_1 = 2.40$, $j_2 = 5.52$ [8, 12], and α_ν real for $\nu < 3$ implies $\lambda/a < 2\pi/5.52$. So, in a tube with diameter $a = 1$ cm, this relation gives $\lambda < 1.13$ cm, the lower bound of the radiofrequency band.

Remark: Chirowave guides work as a mode converter [6] because, due to chirality, the different modes are no more orthogonal according to a relation obtained from the Lorentz reciprocity theorem. So, the last affirmation, that evanescent waves are not important, is at variance if, in a semi-infinite, chiral cylinder, a mode conversion takes also place faster than the decay of evanescent waves.

A particular application of (2) gives the orthogonality properties of Bessel functions [12]:

$$\begin{aligned}\int_0^1 u J_0(j_\nu u) J_0(j_\sigma u)\,\mathrm{d}u &= \frac{1}{2}\delta_{\nu\sigma}[J_1(j_\nu)]^2,\\ \int_0^1 u J_1(l_\nu u) J_1(l_\sigma u)\,\mathrm{d}u &= \frac{1}{2}\delta_{\nu\sigma}\left[J_1'(l_\nu)\right]^2,\end{aligned}$$

where $u = r/a$ and $\delta_{\nu\sigma}$ is the Kronecker symbol.

Applied to the fields (22), (23), these relations imply that the modes in the semi-infinite cylinder are uncoupled, which precludes mode conversion.

5. Post Constitutive Relations in Chiral Cylinders

Fields do not depend on θ. Then, substituting the Post constitutive relations

$$\boldsymbol{D} = \varepsilon\boldsymbol{E} + \mathrm{i}\xi\boldsymbol{B}, \quad \boldsymbol{B} = \mu[\boldsymbol{H} - \mathrm{i}\xi\boldsymbol{E}] \tag{36}$$

into Maxwell's equations (3a), (3b) gives

$$\begin{aligned}&\partial_z E_\theta + \mathrm{i}\omega B_r = 0,\\ &(\partial_r + 1/r)E_\theta - \mathrm{i}\omega B_z = 0,\\ &\partial_z E_r - \partial_r E_z - \mathrm{i}\omega B_\theta = 0,\end{aligned} \tag{37}$$

and

$$\begin{aligned}&1/\mu\partial_z B_\theta - \mathrm{i}\omega\varepsilon E_r + \mathrm{i}\xi\partial_z E_\theta + \omega\xi B_r = 0,\\ &1/\mu(\partial_r + 1/r)B_\theta + \mathrm{i}\omega\varepsilon E_z + \mathrm{i}\xi(\partial_r + 1/r)E_\theta\\ &-\omega\xi B_z = 0,\\ &1/\mu(\partial_z B_r - \partial_r B_z) + \mathrm{i}\omega\varepsilon E_\theta + \mathrm{i}\xi(\partial_z E_r - \partial_r E_z)\\ &-\omega\xi B_\theta = 0.\end{aligned} \tag{38}$$

Taking into account (37), (38) becomes

$$\begin{aligned}&\partial_z B_\theta - \mathrm{i}\omega\varepsilon\mu E_r + 2\omega\xi\mu B_r = 0,\\ &(\partial_r + 1/r)B_\theta + \mathrm{i}\omega\varepsilon\mu E_z - 2\omega\xi\mu B_z = 0,\\ &\partial_z B_r - \partial_r B_z + \mathrm{i}\omega\varepsilon\mu E_\theta - 2\omega\xi\mu B_\theta = 0.\end{aligned} \tag{39}$$

We now introduce the field

$$\underline{\boldsymbol{Q}} = \boldsymbol{B} + \mathrm{i}\gamma\boldsymbol{E}, \quad \gamma = 1 \pm (1+\eta^2)^{1/2}, \quad \eta = \varepsilon/\xi. \tag{40}$$

Then, summing (37) multiplied by $\mathrm{i}\gamma$ to (39) gives

$$\begin{aligned}&\partial_z\underline{Q}_\theta + \omega(2\xi\mu - \gamma)\underline{Q}_r = 0,\\ &(\partial_r + 1/r)\underline{Q}_\theta - \omega(2\xi\mu - \gamma)\underline{Q}_z = 0,\\ &\partial_z\underline{Q}_r - \partial_r\underline{Q}_z - \omega(2\xi\mu - \gamma)\underline{Q}_\theta = 0,\end{aligned} \tag{41}$$

and the $\underline{\boldsymbol{Q}}$-field satisfies the divergence equation

$$(\partial_r + 1/r)\underline{Q}_r + \partial_z\underline{Q}_z = 0. \tag{42}$$

We now get from the third equation of (41)

$$\partial_z{}^2\underline{Q}_r - \partial_r\partial_z\underline{Q}_z - \omega(2\xi\mu - \gamma)\partial_z\underline{Q}_\theta = 0. \tag{43}$$

Substituting (41,1) and (42) into (13) gives the wave equation

$$\left[\partial_r{}^2 + 1/r\partial_r - 1/r^2 + \partial_z{}^2 + \omega^2(2\xi\mu - \gamma)^2\right]\underline{Q}_r = 0. \tag{44}$$

Similarly, applying $(\partial_r + 1/r)$ to (41,3), using (41,2) and (42), we get

$$\left[\partial_r{}^2 + 1/r\partial_r + \partial_z{}^2 + \omega^2(2\xi\mu - \gamma)^2\right]\underline{Q}_z = 0. \tag{45}$$

The wave equations (44), (45) have the following solutions with amplitudes A_r, A_z:

$$\begin{aligned} \underline{Q}_r &= A_r J_1(q_r r)\exp(\mathrm{i}q_z z), \\ \underline{Q}_z &= A_z J_0(q_r r)\exp(\mathrm{i}q_z z), \end{aligned} \tag{46}$$

$$q_r{}^2 + q_z{}^2 = \omega^2(2\xi\mu - \gamma)^2. \tag{46a}$$

Substituting (46) into (41,3) gives the last component $\underline{Q}_\theta$:

$$\omega(2\xi\mu - \gamma)\underline{Q}_\theta = (\mathrm{i}q_z A_r + q_r A_z)\, J_1(q_r r)\exp(\mathrm{i}q_z z), \tag{47}$$

while we get from the divergence equation (42) the following relations between the amplitudes A_r, A_z:

$$q_r A_r + \mathrm{i}q_z A_z = 0. \tag{48}$$

Writing $\boldsymbol{A} = \boldsymbol{M} + \mathrm{i}\boldsymbol{N}$ so that, according to (48), $q_r M_r - q_z N_z = 0$, $q_r N_r + q_z M_z = 0$; then we get

$$\begin{aligned} \{B_r, E_r\} &= \{M_r, N_r\}\, J_1(q_r r)\exp(\mathrm{i}q_z z), \\ \{B_z, E_z\} &= \{M_z, N_z\}\, J_0(q_r r)\exp(\mathrm{i}q_z z), \\ \{B_\theta, E_\theta\} &= \{M_\theta, N_\theta\}\, J_1(q_r r)\exp(\mathrm{i}q_z z), \end{aligned} \tag{49}$$

with

$$\begin{aligned} M_\theta &= [\omega(2\xi\mu - \gamma)]^{-1}(q_r M_z + \mathrm{i}q_z M_r), \\ N_\theta &= [\omega(2\xi\mu - \gamma)]^{-1}(q_r N_z + \mathrm{i}q_z N_r). \end{aligned} \tag{49a}$$

The expressions (49), taking into account (48) and using (36), gives the components of the electromagnetic field in an infinite, chiral medium endowed with the Post constitutive relations.

The wave equations (44), (45) have also as solutions the Fourier-Bessel expansions (22), (23) with $\boldsymbol{Q}$ changed into the vector field $\underline{\boldsymbol{Q}}$ (40), while j_ν, l_ν being the positive zeroes of J_0, J_1 and the parameters α_ν, β_ν become

$$\begin{aligned} \beta_\nu &= \left(\sigma^2 - l_\nu{}^2/a^2\right)^{1/2}, \\ \alpha_\nu &= \left(\sigma^2 - j_\nu{}^2/a^2\right)^{1/2}, \\ \sigma^2 &= \omega^2(2\xi\mu - \gamma)^2. \end{aligned} \tag{50}$$

Using the first two equations of the sets (37), (39) and the divergence equation (42), it is easy to obtain the solutions of Maxwell's equations satisfying the boundary conditions (24) on the surface $r = a$ of the cylinder.

Explicitly, taking into account the properties (18) of the Bessel functions J_0, J_1, we get from (37,1), (37,2) and (42) for the components $B_{r,\nu}$, $B_{z,\nu}$, $E_{\theta,\nu}$, of the ν-th mode:

$$\begin{aligned} B_{r,\nu} &= A_{r,\nu} J_1(l_\nu r/a)\exp(\mathrm{i}\beta_\nu z), \\ B_{z,\nu} &= -\mathrm{i}(l_\nu/a\beta_\nu)\, A_{r,\nu} J_0(l_\nu r/a)\exp(\mathrm{i}\beta_\nu z), \end{aligned} \tag{51,1}$$

$$E_{\theta,\nu} = -(\omega/\beta_\nu)A_{r,\nu} J_1(l_\nu r/a)\exp(\mathrm{i}\beta_\nu z), \tag{51,2}$$

and $B_{r,\nu}(a,z) = 0$, $E_{\theta,\nu}(a,z) = 0$.

For the three other components $E_{z,\nu}$, $E_{r,\nu}$, $B_{\theta,\nu}$, of the ν-th mode satisfying (39,1), (39,2) and (42), it comes

$$\begin{aligned} E_{z,\nu} &= \mathrm{i}/\alpha_\nu A_{z,\nu} J_0(j_\nu r/a)\exp(\mathrm{i}\alpha_\nu z), \\ E_{r,\nu} &= (a/j_\nu)\, A_{z,\nu} J_1(j_\nu r/a)\exp(\mathrm{i}\alpha_\nu z), \end{aligned} \tag{52,1}$$

$$\begin{aligned} \mu^{-1}B_{\theta,\nu} &= (\omega\varepsilon a/j_\nu\alpha_\nu)A_{z,\nu} J_1(j_\nu r/a)\exp(\mathrm{i}\alpha_\nu z) \\ &\quad + 2\mathrm{i}\omega(\xi l_\nu/a\beta_\nu)A_{r,\nu} J_0(l_\nu r/a)\exp(\mathrm{i}\beta_\nu z), \end{aligned} \tag{52,2}$$

and $E_{z,\nu}(a,z) = 0$.

From there, to analyze the electromagnetic propagation in a semi-infinite cylinder, one would proceed as with the Tellegen constitutive relations by matching, on the base $z = 0$ of the cylinder, the components (33) of the incident field with the infinite sums $\sum_\nu E_{\theta,\nu}$, $\sum_\nu E_{z,\nu}$, obtained from (51,2) and (52,1).

6. Conclusions

1. The Fourier-Bessel representations of electromagnetic fields appear as a suitable tool to handle wave propagation in semi-infinite structures with cylindrical symmetry. Harmonic fields, that do not depend on the angle θ, only need the Bessel functions J_0, J_1. Otherwise for the m-th mode with the phase factor $\exp(\mathrm{i}m\theta)$, the Bessel functions J_m and J_{m+1} are required as shown by the presence of the derivative $1/r^2\partial_\theta{}^2$ in the wave equations (8a), (9a), (10a): calculations become a bit more intricate and have still to be performed.

Applications of this theoretical work, for instance, to wave propagation in rods, will require algorithms to be devised through some of the well known numerical methods suited to this kind of problems.

2. It is stated in the introduction that the Post constitutive relations follow from the covariance of the electromagnetism under the Lorentz group, and it has been pointed out [13] that they are valid for a chiral medium

composed of a dilute suspension of perfectly conducting single-turn helices embedded in an otherwise achiral host medium.

A comparison of the formalisms displayed in Sections 2 – 4 and in Section 5 shows that the same kind of calculations are required with Tellegen and Post constitutive relations. But, there is an important difference: the $\underline{\boldsymbol{Q}}$-field (40) is a self-dual tensor of the type introduced by Silberstein and Bateman [14]. More exactly, let $F_{\mu\nu}$, $\mu, \nu = 0, 1, 2, 3$ be the electromagnetic field tensor depending on $\boldsymbol{E}$ and $\boldsymbol{B}$ [15], then the $\boldsymbol{Q}$-field is the tensor

$$F_{\mu\nu} + \mathrm{i}\gamma/2\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}, \tag{53}$$

in which $\varepsilon_{\mu\nu\alpha\beta}$ is the totally antisymmetric fourth rank tensor, $\mu, \nu, \alpha, \beta = 0, 1, 2, 3$ with summation on the repeated indices. At the opposite, the $\boldsymbol{Q}$-field introduced in Section 2.3 has no definite tensorial character. Consequently, sticking to the relativistic nature of electromagnetism only the solutions of Maxwell's equations obtained in Section 5 have a physical meaning. Since they do not depend on its structure and since the expressions (36) valid in isotropic media are easily generalized to anisotropic ones [11], Post's constitutive relations could be considered as ''the" constitutive relations in any chiral material. This is an important result with the increasing development of new chiral materials specially in nanotechnology [1 – 3].

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